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ANALYSIS OF WATER HAMMER BY CHARACTERISTICS

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HYDRAULICS DIVISION

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PAPERS

ANALYSIS OF WATER HAMMER BY CHARACTERISTICS

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Note.—Written comments are invited for publication; the last discussion should be submitted by?
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Synopsis

In the analysis of water-hammer problems the effects of friction, kinetic energy, and change in wave form may be negligible from the practical point of view. However, it is worthwhile to develop a general method of study that makes allowance for these factors, so that it can be seen what order of approximation is involved in neglecting them.

This paper presents a solution using the characteristics, or characteristic directions, of the equations that are involved in the water-hammer problem. The method evolved lends itself to a graphical treatment in some cases. The development of the method rather that the consideration of problems of practical significance is the concern of this paper.

Notation.—The letter symbols introduced in this paper are defined where they first appear, in the text, or by illustration, and are assembled alphabetically in Appendix II for convenience of reference.

The analysis of water hammer problems in hydraulic systems has been based primarily on the wave equation

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} \qquad (1)$$

in which h is the head on the system, x is a length measured along the pipe, c equals the speed of surge, and t denotes time. Elegant solutions have been provided by Robert W. Angus[2,3,4] and G. R. Rich,[5] the former giving a graphical

[2] "Simple Graphical Solution for Pressure Rise in Pipes and Pump Discharge Lines," by Robert W. Angus, *The Engineering Journal*, Vol. 18, 1935, p. 72.
[3] "Water-Hammer Pressures in Compound and Branched Pipes," by Robert W. Angus, *Transactions*, ASCE, Vol. 104, 1939, p. 340.
[4] "Water Hammer in Pipes, Including Those Supplied by Centrifugal Pumps: Graphical Treatment," by Robert W. Angus, *Proceedings*, Inst. of Mech. Engrs., Vol. 136, 1937, p. 245.
[5] "Water-Hammer Analysis by the Laplace-Mellin Transformation," by G. R. Rich, *Transactions*, ASME, Vol. 67, 1945, p. 361.

solution and the latter an analytical solution based on the Laplace-Mellin transformation. However, in neither case is it possible to incorporate the full effects of friction or kinetic energy, although an approximate correction for these effects has been given by Mr. Angus. Mr. Rich includes the effects of linear friction but neglects kinetic energy entirely, since his solution can deal only with linear quantities. In neither case can the effects of change in wave velocity be considered.

General Theory for Ideal Conditions

The usual form of the equations of motion, neglecting viscosity, kinetic energy, and change in wave form, are

$$\frac{\partial u}{\partial x} = \frac{g}{c^2}\frac{\partial h}{\partial t} \quad \ldots\ldots (2a)$$

$$\frac{\partial u}{\partial t} = g\frac{\partial h}{\partial x} \quad \ldots\ldots (2b)$$

in which u is the velocity of the fluid and g denotes the acceleration due to gravity. For convenience, in Eqs. 2 the velocity u is measured in the opposite direction to that in which x is measured.

If Eq. 2*a* is multiplied by λ_1, Eq. 2*b* is multiplied by λ_2 and the two are added, there results

$$\lambda_1 \frac{c^2}{g}\frac{\partial u}{\partial x} + \lambda_2 \frac{\partial u}{\partial t} - \lambda_1 \frac{\partial h}{\partial t} - \lambda_2 g \frac{\partial h}{\partial x} = 0 \quad \ldots\ldots (3)$$

If there is some curve in the xt-plane (Fig. 1), such that

$$\frac{dx}{dt} = \frac{\lambda_1}{\lambda_2}\frac{c^2}{g} = \frac{\lambda_2}{\lambda_1} g \quad \ldots\ldots (4)$$

then at any point on this curve $\dfrac{\lambda_1}{\lambda_2} = \pm \dfrac{g}{c}$. Thus through any point in the xt-plane there are two directions given by $\dfrac{dx}{dt} = \pm c$. If c is a constant, these directions define two straight lines, denoted by C_{pos} and C_{neg}, called the characteristics of Eqs. 2. For C_{pos} the equation of the line is

$$x + c\,t = \text{constant} \quad \ldots\ldots (5a)$$

and for C_{neg} the equation is

$$x - c\,t = \text{constant} \quad \ldots\ldots (5b)$$

Through every point in the xt-plane passes a C_{pos}-curve and a C_{neg}-curve. Consequently, each point in the xt-plane can be identified by the two curves passing through it.

Now, consider the changes in u and h along a C_{pos}-curve. From Eq. 3,

$$\frac{\lambda_1 c^2}{g}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}\frac{dt}{dx}\right)-\lambda_2 g\left(\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t}\frac{dt}{dx}\right)=0 \quad \ldots\ldots (6a)$$

from which

$$\frac{c}{g}\frac{du}{dx}-\frac{dh}{dx}=0 \quad \ldots\ldots (6b)$$

in which $\frac{du}{dx}$ and $\frac{dh}{dx}$ are, respectively, the rate of change of u and h with regard to x along the C_{pos}-curve.

Alternatively,

$$\lambda_2\left(\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial t}\right)-\lambda_1\left(\frac{\partial h}{\partial x}\frac{dx}{dt}+\frac{\partial h}{\partial t}\right)=0 \quad \ldots\ldots (7a)$$

from which

$$\frac{c}{g}\frac{du}{dt}-\frac{dh}{dt}=0 \quad \ldots\ldots (7b)$$

Therefore, from Eqs. 6*b* and Eq. 7*b*, the total change Δu in u and the total change Δh in h along the C_{pos}-curve are related by

$$\frac{c}{g}\Delta u=\Delta h \quad \ldots\ldots (8a)$$

and along the C_{neg}-curve, they are related by

$$\frac{c}{g}\Delta u=-\Delta h \quad \ldots\ldots (8b)$$

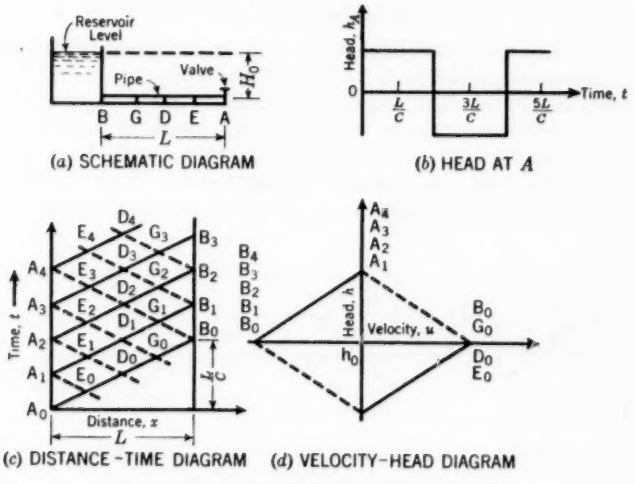


FIG. 1.—SUDDEN CLOSURE, IDEAL CONDITIONS

As an example, consider the sudden closure of the valve at the end of the pipe leading from the reservoir in Fig. 1(*a*).

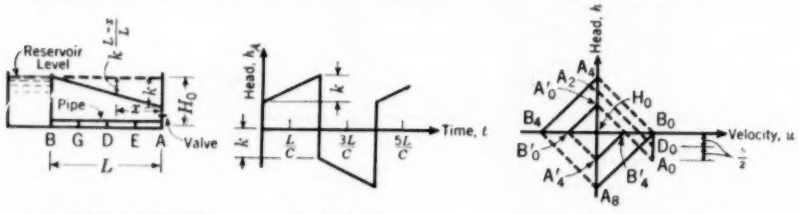
The distance-time diagram is shown in Fig. 1(*c*), and passing through each point there are two lines with slopes to the x-axis of $\pm 1/c$. The C_{pos}-curves are shown by solid lines and the dashed lines represent the C_{neg}-curves. Nothing occurs at any point along the pipe line AB until the first wave reaches that point. Thus, all the points on the line A_0B_0 in Fig. 1(*c*) have their initial u-value and h-value. This fact and the fact that the head at point B remains constant constitute the boundary data of the problem.

The velocity-head diagram is shown in Fig. 1(*d*). The points E_0, D_0, G_0, and B_0 are located by the use of the initial conditions of the problem. The points are coincident on the u-axis since all losses are neglected.

For the typical C_{neg}-characteristic, from Eq. 8*b*, since c is a constant, $\frac{dh}{du} = -\frac{c}{g}$. However, the velocity at A is zero for all times after the valve is closed. Hence, if a line is drawn through point E_0, in the uh-diagram, having a slope of $-\frac{c}{g}$, so as to intersect the h-axis, this intersection is A_1. All the points, A_1, A_2, A_3, and A_4, corresponding to times at the valve of $L/2c$, L/c, $3L/2c$, and $2L/c$, respectively, will coincide, since the lines E_0A_1, D_0A_2, and so on, in the uh-diagram have the same slopes and pass through a common point.

The B-points on the uh-diagram are obtained from the C_{pos}-curves drawn through the A-points. Since for the C_{pos}-characteristic $\frac{dh}{du} = \frac{c}{g}$, and at point B the head is always equal to H_0, the points B_1, B_2, B_3, and B_4 in the uh-plane are obtained by drawing lines through the corresponding A-points, having a slope of $\frac{c}{g}$ to the u-axis, and intercepting it at points B_1, B_2, B_3, and B_4. The remainder of the diagram is then completed as shown in Fig. 1(*d*) in which the solid and dashed lines correspond to those of Fig. 1(*c*).

However, the initial distribution of head along the pipe is not constant. In Fig. 2 the previous example is investigated assuming an initially linear distribution of head, as shown in Fig. 2(*a*). In this case the head on the pipe is H_0, and the head decreases linearly throughout the length of the pipe, by an amount γ. Under these conditions, the distance-time diagram is similar



(*a*) SCHEMATIC DIAGRAM (*b*) HEAD AT *A* (*c*) VELOCITY-HEAD DIAGRAM

FIG. 2.—SUDDEN CLOSURE, NEGLECTING DISSIPATION AND KINETIC ENERGY

to Fig. 1(*c*), but the initial positions of A_0, E_0, D_0, and G_0 in the uh-plane are no longer on the u-axis. The modification is shown in Fig. 2(*c*). It is of interest to note that the wave form of the head at point A, shown in Fig. 2(*b*), is not flat-topped as in Fig. 1(*b*), but rises linearly over the half period.

Analysis with Friction Included

Neglecting the energy terms, the equations of motion become

$$\frac{\partial u}{\partial t} - g\frac{\partial h}{\partial x} + \beta u^n = 0 \quad \ldots\ldots (9a)$$

and

$$\frac{c^2}{g}\frac{\partial u}{\partial x} - \frac{\partial h}{\partial t} = 0 \quad \ldots\ldots (9b)$$

in which the friction term is of the form βu^n, with β a friction term coefficient and n an exponent, being constants.

As shown previously, the characteristics are given by the equations $\frac{dx}{dt} = \pm c$, and the total variations Δu and Δh of u and h along the C_{pos}-curve are related by

$$\frac{c}{g}\Delta u - \Delta h + \frac{\beta c}{g} u^n \Delta t = 0 \quad \ldots\ldots (10a)$$

and along the C_{neg}-curve they are related by

$$\frac{c}{g}\Delta u + \Delta h + \frac{\beta c}{g} u^n \Delta t = 0 \quad \ldots\ldots (10b)$$

These equations can be used to obtain a step-by-step solution.

Case n = 1.—However, an alternate form to Eqs. 10 can be obtained if $n = 1$, and $u' = e^{\beta t} u$. Then Eqs. 10 become for the C_{pos}-curve—

$$\frac{c}{g} e^{-\beta t} \Delta u' = \Delta h \quad \ldots\ldots (11a)$$

and for the C_{neg}-curve—

$$\frac{c}{g} e^{-\beta t} \Delta u' = -\Delta h \quad \ldots\ldots (11b)$$

For graphical solutions it is convenient to use dimensionless quantities. Therefore, let $u'_1 = \frac{u'}{u_0}$, $h_1 = \frac{g h}{c u_0}$, $\beta_1 = \frac{\beta L}{c}$, and $\tau = \frac{c t}{L}$, in which L is the length of pipe and u_0 is the initial velocity of the fluid. Then for the C_{pos}-curve, Eq. 11*a* becomes

$$e^{-\beta \tau} \Delta u'_1 = \Delta h_1 \quad \ldots\ldots (12a)$$

and for the C_{neg}-curve, Eq. 11*b* becomes

$$e^{-\beta \tau} \Delta u'_1 = -\Delta h_1 \quad \ldots\ldots (12b)$$

Since the friction term should be of the form βu^2, Eqs. 10 will be developed in a form suitable for graphical solution with $n = 2$.

Case $n = 2$.—As for the case of $n = 1$, a dimensionless form is most suitable. Therefore, let $u_1 = \frac{u}{u_0}$, $h_1 = \frac{g h}{c u_0}$, $\tau = \frac{c t}{L}$, and $\beta'_1 = \beta u_0 \frac{L}{c}$. Then for the C_{pos}-curve—

$$\Delta u_1 - \Delta h_1 + \beta'_1 u^2_1 \Delta \tau = 0 \ldots\ldots\ldots\ldots\ldots\ldots (13a)$$

and for the C_{neg}-curve—

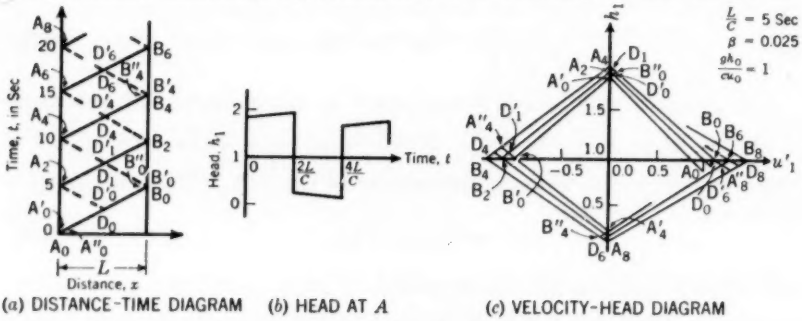
$$\Delta u_1 + \Delta h_1 + \beta'_1 u^2_1 \Delta \tau = 0 \ldots\ldots\ldots\ldots\ldots\ldots (13b)$$

with the restriction that reversal of flow will change the sign of the friction term.

The Effects of Friction

To illustrate the effects of friction, consider the case of $n = 1$, shown in Fig. 3, which represents a sudden closure of the valve at point A in the pipe line previously considered. Assuming initially a head loss of $0.125 \frac{c u_0}{g}$ distributed linearly from reservoir to valve, and since $u_1 = e^{\beta t} \frac{u}{u_0}$ points A_0, E_0, D_0, G_0, and β_0 can be located in the uh-plane, shown in Fig. 3(*c*). From Eqs. 12, the slope $\frac{dh_1}{du_1}$ is known as a function of the time along the C_{pos} and C_{neg}-curves. If these curves are so flat that they can be approximated as a series of straight lines, then the slope of the curves $\frac{dh_1}{du_1} = \pm e^{-\beta\tau}$ in which τ is the average time over the interval. For ease in graphical representation a comparatively large time interval of 2.5 sec is assumed.

Considering the point D_0 in Fig. 3(*a*)—D'_0 is located very close to point D_0, in time, along the C_{neg}-characteristic D_0A_2. Over this short period of time the head and velocity are very great. At D'_0, the first compression wave traveling up the pipe has passed a point $L/2$ from the valve. In order to use the previously mentioned approximation, it is necessary to find the points



(*a*) DISTANCE-TIME DIAGRAM (*b*) HEAD AT *A* (*c*) VELOCITY-HEAD DIAGRAM

Fig. 3.—Sudden Closure, Linear Friction

A'_0, D'_0, and so on.

Consider the point A''_0 along the C_{pos}-characteristic A_0B_0, very close to A_0. In Fig. 3(*c*) this is a point very close to A_0 and at the limit will coincide with A_0. To obtain A'_0, the C_{neg}-characteristic is drawn through A''_0 in the xt-diagram to intersect the t-axis. Since this time interval is so short the slope in the $u'_1\,h_1$-diagram is $\dfrac{dh_1}{du_1} = -1$.

Therefore, through point A_0 a line is drawn, inclined towards the u'_1 axis at an angle of 135°, which intersects the h_1 axis at A'_0. To obtain D'_0 it is noted that it lies at the intersection of the C_{pos}-characteristic $A'_0B''_0$ and the C_{neg}-characteristic D_0A_2. Hence, in the $u'_1\,h_1$-plane, through A'_0 a line is drawn having a slope of $e^{-\beta\,2.5+0/2}$ and through D_0 a line is drawn having a slope of $-e^{-2.5\beta}$. The intersection of these lines locates the point D'_0. In a similar manner the head and velocity are obtained at all points along the characteristic B_0A_4. These can be used to extend the solution to the characteristic A_4B_4, and then, in turn, the information on A_4B_4 can be used to carry the solution as far as required, in exactly the same manner as the initial data were used along the characteristic A_0B_0.

The wave form for this case is shown in Fig. 3(*b*). The wave form shows a decay with time, as is expected. The final result may be compared with the analytic solution of the same problem obtained by Mr. Rich.[5]

Since Mr. Rich's operational solution can not be used if a nonlinear velocity-head relationship at point A is given, the previous problem is considered in Fig. 4 and Fig. 5, in which the velocity and head at point A are assumed to be related by an equation of the form $\dfrac{u_A}{u_0} = k\sqrt{\dfrac{h_A}{h_0}}$ for all time, in which u_A is the velocity at point A and h_A is the head at point A.

Fig. 4 shows the solution of the problem when the valve at point A closes in such a manner that k varies linearly from 1 to 0 in 12 sec, assuming a linear friction term. Fig. 5 considers the same problem but assumes a friction term of the form $\beta\,u^2$. The solution given in Fig. 4 follows the procedure of Fig. 3. However, some modification occurs in the solution presented in Fig. 5. Eqs. 13 apply to this case, and u_1 and h_1 are given along the C_{pos}-characteristic A_0B_0.

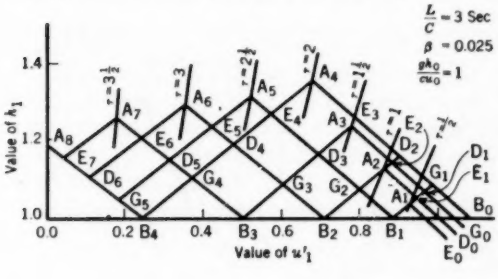


Fig. 4.—Linear Closure, Linear Friction

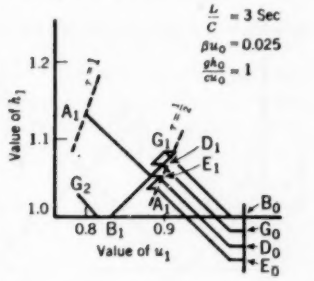


Fig. 5.—Linear Closure, $\beta\,u^2$ Friction

Since the valve is closed continuously no abrupt changes occur in u_1 and h_1 due to the passage of the surge. Thus the location of the points A'_0, E'_0, and so on, is obviated. This also applies to Fig. 4. In Fig. 5 the intervals of time taken are $t = L/4\,c$. It is assumed that over this period of time the term $\beta'_1\,u^2_1$ is sensibly constant and may be represented by its average value. If excessive variation occurs, as will be obvious in the solution, smaller increments of time must be taken.

In the velocity-head diagram, Fig. 5, the points E_0, D_0, G_0, and B_0 are located by using the initial conditions. Then A_1 is located in the $u_1\,h_1$-plane. To obtain the average value of u_1 over the interval a first approximation is made by assuming no friction, as in Fig. 6. From Fig. 6 $(u_1)_{ave} = 0.96$ so that

$$0.075\ (u^2_1)_{ave}\,\Delta\tau = 0.075 \times 0.96^2 \times \frac{1}{4} = 0.017.$$

Since the increment Δu_1 is negative, $\Delta h_1 = |\,\Delta u_1\,| - 0.017$ in which Δh_1 is the increment of h_1 in going from E_0 to A_1 along the $C_{+\varepsilon}$-characteristics E_0A_1. Therefore, from a point 0.017 horizontally to the left of E_0 in the $u_1\,h_1$-diagram a line is drawn at an angle of 135° to the horizontal. This line intersects the parabola $\tau = \frac{1}{2}$ at A_1. The same procedure is used for each element of the characteristic diagram.

Fig. 7 shows a comparison between the two forms of friction correction. The dimensionless head at the valve is plotted against time, using values of β in each case, to give the same head loss from reservoir to valve under the initial steady conditions. For the problem considered it is seen that up to the maximum head there is a negligible difference. If no friction is assumed, the result obtained is only 4% in excess of the values obtained assuming friction.

Including the Effect of Kinetic Energy and Friction and Neglecting the Change in Wave Form.—The general equations of motion are

$$u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} + \beta\,u^n = 0 \qquad (14)$$

and

$$\frac{c^2}{g}\frac{\partial u}{\partial x} - \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = 0 \qquad (15)$$

in which u is measured in the direction opposite to that in which x is measured.

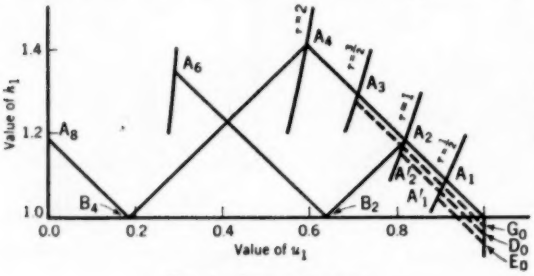


FIG. 6.—LINEAR CLOSURE

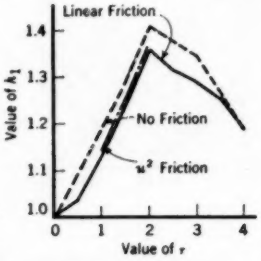


FIG. 7.—LINEAR CLOSURE, HEAD AT A

As is done previously, Eq. 14 is multiplied by λ_1 and Eq. 15 is multiplied by λ_2, choosing λ_1/λ_2 so that $\lambda_1/\lambda_2 = \pm c/g$. The characteristics are given by

$$\frac{dx}{dt} = -(u \pm c) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

and for the C_{pos}-characteristic—

$$\frac{c}{g}\Delta u - \Delta h + \frac{\beta c}{g} u^n \Delta t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (17a)$$

and for the C_{neg}-characteristic—

$$\frac{c}{g}\Delta u + \Delta h + \frac{\beta c}{g} u^n \Delta t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (17b)$$

Let $u_1 = \frac{u}{u_0}$, $h_1 = \frac{g h}{c u_0}$, $\beta_2 = \beta u^{n-1}{}_0 \frac{L}{c}$, and $\tau = \frac{c t}{L}$. Then for the C_{pos}-curve, Eq. 17*a* becomes

$$\Delta u_1 - \Delta h_1 + \beta_2 u^n{}_1 \Delta\tau = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (18a)$$

and for the C_{neg}-curve, Eq. 17*b* becomes

$$\Delta u_1 + \Delta h_1 + \beta_2 u^n{}_1 \Delta\tau = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (18b)$$

The characteristics, given by Eqs. 17, are curved and the solution will require these curves to be determined in the xt-plane, simultaneously with the uh-curve in the uh-plane. However no great difficulty is entailed, although a graphical solution is not possible.

Before indicating a solution, a few observations can be made. In dimensionless form Eq. 16 becomes,

$$\frac{dx_1}{d\tau} = -\left(u_1 \frac{u_0}{c} \pm 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

in which $x_1 = \frac{x}{L}$, $\tau = \frac{c t}{L}$, and $u_1 = \frac{u}{u_0}$. For closure of the valve, u_1 is always less than 1, where as generally u_0/c is of the order of 10^{-3}, so that the correction is of the order of 10^{-3}, and thus will be negligible. Therefore it appears that the kinetic energy can be safely neglected where conditions at the valve are not extremely sensitive to time since, considering only conditions at the valve, the curving of the characteristics can only affect time. This is shown in the following example.

Consider once again, the pipe discharging from a reservoir with a valve at the end, for which time of closure is 12 sec, $\frac{L}{c} = 3$ sec, $u_0 = 16$ ft per sec, and $c = 4{,}000$ ft per sec. A closure relation of the form $\frac{u_A}{u_0} = k\sqrt{h_A/h_0}$ is assumed in which k varies linearly with the time.

Then for the C_{pos}-characteristics $\Delta x_1 = -(1 + 0.004\, u_1)\Delta\tau$ and for the

C_{neg}-characteristics $\Delta x_1 = (1 - 0.004\ u_1)\ \Delta\tau$. Using values from Fig. 5 as a first approximation, $(u_1)_{avg} = 0.96$ for the time interval E_0A_1. Hence, $\Delta\tau = \frac{1}{4}(1 - 0.00384) = 0.24904$ which gives the time corresponding to point A_1, in the $x_1\ \tau$-diagram. Thus in the $u_1\ h_1$-diagram the parabola on which A_1 lies must be drawn for $\tau = 0.24904$. The next point located is E_1. Both time and heads are known at A_1 and D_0 and assuming the characteristics A_1E_1 and D_0E_1 are to be straight lines, two equations are obtained to compute τ and h at E_1. From Fig. 8, a general formula can be derived that is convenient when making these computations. From Fig. 8

$$\Delta\tau = \frac{k_1 - k_2 \cot\theta_1}{\cot\theta_1 + \cot\theta_2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20a)$$

and

$$\Delta x_1 = (k_2 + \Delta\tau)\cot\theta_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20b)$$

From Eqs. 20 E_1, $\cot\theta_1 = 0.9964$, and $\cot\theta_2 = 1.0036$, in which the estimated value of u_1 has been obtained from Fig. 5.

Hence, $\Delta\tau = 0.2495$, giving τ at E_1 equal to 0.7495. Consequently, for A_2, $\tau = \tau$ at $E_1 + \Delta\tau$ and $\Delta\tau = \frac{1}{4}(1 - 0.004 \times 0.85)$, which gives τ at A_2 equal to 0.9986. In a similar manner the coordinates of any point in the $x_1\ \tau$—diagram can be computed. The method of construction of the $u_1\ h_1$-diagram is unaltered.

Verification of Computations

Since the use of the step-by-step method of solution will introduce cumulative errors, it is necessary to be able to check the error as the solution progresses. Referring to Eqs. 18, for the C_{pos}-curve—

$$(u_1)_G - (h_1)_G = (u_1)_F - (h_1)_F - \int_F^G \beta_1\ u^n{}_1\ d\tau \ldots\ldots\ldots (21a)$$

and for the C_{neg}-curve

$$(u_1)_G - (h_1)_G = (u_1)_F + (h_1)_F - \int_F^G \beta_1\ u^n{}_1\ d\tau; \ldots\ldots\ldots (21b)$$

n which F and G are two points on the characteristic. From the step-by-step

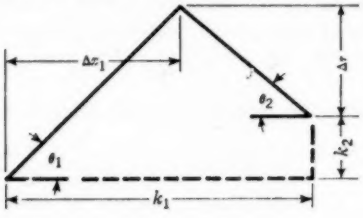


Fig. 8.—General Case

solution there is a value for u_1 at every point on a given characteristic. Thus the integrals in Eqs. 21 can be determined and, hence, values of h and u at G should agree with those obtained from the step-by-step solution.

Conclusion

From the illustrations it can be seen that a complete solution of the analysis of surges in pipe lines can be made when the conception of the characteristics of the equations of motion is introduced. In the paper two types of friction have been considered merely to illustrate the method—any form of friction as a function of the velocity could be used. However, notwithstanding existing methods of correcting for friction, it is clear that no steady-state form of friction correction is exact, although satisfactory approximations based on them may be made. From the examples considered, it can be seen that, although the friction factors considered on a steady-state basis are very great, they have a negligible effect on the maximum heads. This result may not apply to complex systems because of the retardation introduced. The kinetic energy correction, however, would seem to have been made incorrectly by previous investigators. As the paper shows, the correct basis for a kinetic energy correction is the ratio of the fluid velocity to the surge velocity. This is because only in exceptional cases will appreciable curving of the characteristics in the xt-plane occur. Thus, in general, velocity-head corrections are negligible.

No great difficulty should be found in applying the method to any case, no matter how complex. The procedure becomes apparent if the ideal case is taken as a guide, and for this the papers by Mr. Angus are available.[2,3,4]

APPENDIX I. ONE DIMENSIONAL FLOW

Derivation of Equations.—If x is the direction of flow of a compressible fluid and u is measured in the same direction, then the Navier-Stokes equation for the flow of an ideal compressible fluid becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \frac{g}{\rho}\frac{d}{dx}(\rho) \quad \ldots\ldots\ldots\ldots\ldots (22)$$

in which p is the pressure on the fluid and the z-axis is measured vertically downward and makes an angle α with the direction of flow.

If all the pressure changes are considered from hydrostatic and $p' = p - \rho g z$, in which z is the distance from the z-axis, then Eq. 22 becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

By definition, if Δp is the change in pressure producing a change ΔV in volume of fluid V then

$$\Delta p = -K\frac{\Delta V}{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

in which K is the bulk modulus of the fluid.

However, since mass $= \rho V$, then $\frac{\Delta\rho}{\rho} = -\frac{\Delta V}{V}$ Eq. 24 becomes

$$\Delta p = K \frac{\Delta\rho}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

in which ρ is the density of the fluid. From Eq. 25,

$$\frac{\partial p'}{\partial x} = \frac{K}{\rho}\frac{\partial \rho}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26a)$$

and

$$\frac{\partial p'}{\partial t} = \frac{K}{\rho}\frac{\partial \rho}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26b)$$

It is customary to express pressures in terms of the head of the flowing fluid, so by letting $p' = \rho\, g\, h$ then,

$$\frac{\partial p'}{\partial x} = g\left(\rho\frac{\partial h}{\partial x} + h\frac{\partial \rho}{\partial x}\right) = g\left(\rho\frac{\partial h}{\partial x} + \frac{\rho\, h}{K}\frac{\partial p'}{\partial x}\right) \quad \ldots\ldots\ldots (27a)$$

from which

$$\frac{\partial p'}{\partial x} = \rho\frac{g\dfrac{\partial h}{\partial x}}{1 - \rho\dfrac{g\,h}{K}} \approx \rho\, g\frac{\partial h}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27b)$$

since $K \gg \rho\, g\, h$.

In a similar manner

$$\frac{\partial p'}{\partial t} = \rho\, g\frac{\partial h}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Substituting Eq. 27*b* into Eq. 23,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\,g\frac{\partial h}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

The remaining equation required for the solution is given by the condition of continuity, which under the prescribed conditions takes the form:

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

By using Eqs. 26, 27*b*, and 28, Eq. 30 becomes

$$\frac{\partial u}{\partial x} = -\frac{\rho\, g}{K}\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Eq. 29 and Eq. 31 are the equations governing the flow. As presented

they are nonlinear. The usual procedure is to linearize them and to treat $C^2 = K/\rho$ as a constant. This is similar to neglecting the terms $u\frac{\partial u}{\partial x}$ and $u\frac{\partial h}{\partial x}$. Under these conditions, Eq. 29 and Eq. 31 become

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32a)$$

and

$$\frac{\partial u}{\partial x} = -\frac{g}{c^2}\frac{\partial h}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32b)$$

With c as a constant, Eqs. 32 are further reduced to

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33a)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33b)$$

Effect of Friction.—If viscous effects are to be considered, an additional term $\nu\,\nabla^2 u$ must be added to the right side of Eq. 22. For steady one-dimensional viscous flow $\nu\,\nabla^2 u$ is equal to $-\beta u$, in which β is a constant. Since $-\beta u$ is linear, it is the form usually adopted and added to the right side of Eq. 29.

It is often suggested that the allowance for friction should be made following the same form as the shock or eddy-loss developed for steady flow. In this case the term $\frac{f u^2}{2d} = \beta u^2$ is added to the right side of Eq. 29 (in which f is the friction factor and d is the diameter of the pipe), with the stipulation that under all conditions it will be effective in a direction opposing the velocity of flow.

APPENDIX II. LIST OF SYMBOLS

The following letter symbols, adopted for use in this paper and its discussion, conform essentially with American Standard Letter Symbols for Hydraulics (ASA-Z10.2—1942), prepared by a committee of the American Standards Association, with ASCE participation and approved by the Association in 1942;

C_{po} and C_{neg} = characteristics of the equations of motion;
c = speed of surge;
d = pipe diameter;
f = friction factor;
g = acceleration of gravity;
H_0 = head in the reservoir;
h = head of the system;

h_1 = dimensionless head = $\frac{g\,h}{c\,u_0}$;

K = bulk modulus;

k = effective coefficient of discharge at the valve;

L = length of pipe;

n = an exponent;

t = unit of time;

u = velocity of the fluid;

u_1 = dimensionless velocity = $\frac{u}{u_0}$;

x = distance measured along the pipe;

x_1 = dimensionless length = $\frac{x}{L}$;

z = distance from the x-axis;

β = a friction term coefficient;

γ = loss in head as a result of friction;

Δ = the total change in the quantity for the period under consideration;

λ = a number such that $\frac{\lambda_1}{\lambda_2} = \pm\frac{g}{c}$;

ρ = density of the fluid; and

τ = dimensionless time = $\frac{c\,t}{L}$.

(Subscripts 0, 1, 2, 3, . . . are used to indicate conditions along the pipe at successive time increments and the prime superscripts denote differential displacements of the points of study.)